CZECH TECHNICAL UNIVERSITY IN PRAGUE

Robotics: Rigid body motion

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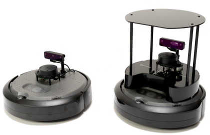
23.09.2023

What is robot?



Mobilní robot, UGV -
unmanned ground vehicle

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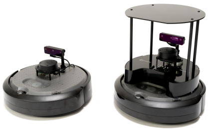

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Flying robots (*e.g.* drones)

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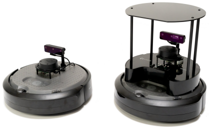

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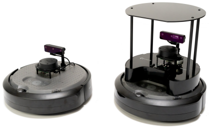


Mobilní robot, UGV - unmanned ground vehicle

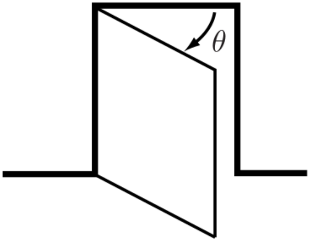
Flying robots (*e.g.* drones)

Walking robots (*e.g.* humanoids)

Manipulators (např. Franka Emika Panda)

Robot configuration

- Complete specification of the position of every point of the robot.



The configuration is described by the angle θ.

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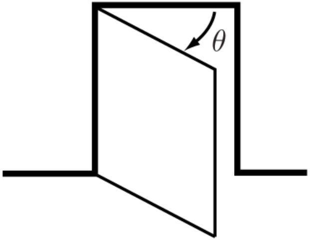

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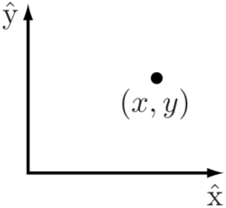

Point in plane is described by two coordinates.

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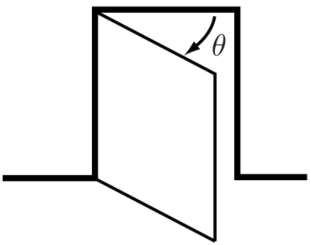

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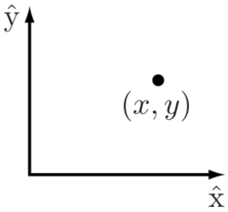

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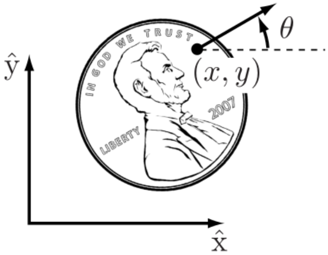

Planar rigid object configuration consists of the position and orientation.

Degrees of freedom (DoF)

- The minimum number of real-valued coordinates needed to represent the configuration.
  - door: 1
  - planar point: 2
  - planar rigid object: 3
  - manipulators: from 1 (*e.g.* rotating table) to tens (*e.g.* humanoids)

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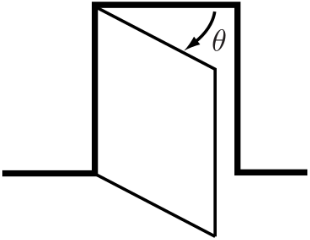
- Determining DoF
  - (sum of freedom of the points) - (number of independent constraints)
  - Rigid objects - how it is defined?
    - The distance between any two given points on a rigid body remains constant
    - Exercise: write constraints for N points of planar rigid object
  - For some robots, determining number of DoF is non-trivial

Configuration space - $\mathcal{C}$

- The N-dimensional space (N correspond to number of DoF)
- Every point of configuration space correspond to one configuration
- Contains all possible configurations of the robot



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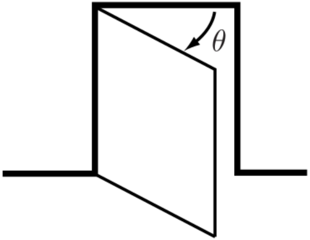

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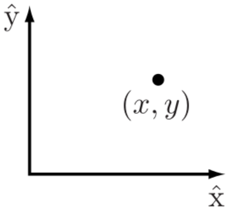

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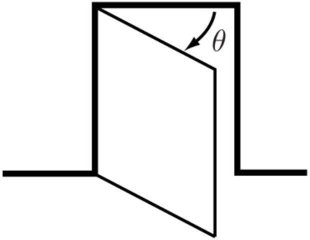

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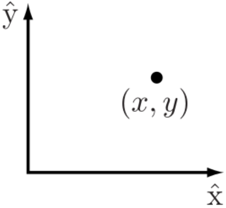

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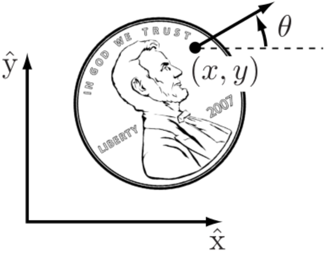
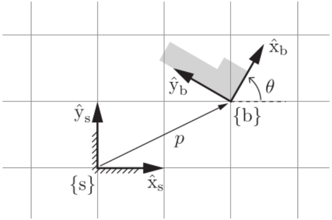


$\mathcal{C} : \mathbb{R}^2 \times \langle 0°, 360°)$ why not $\mathbb{R}^3$?

Rigid body motion in plane



- We attach a **body** frame to rigid body
  - Usually placed in the center of mass (but not required)
  - Can be placed outside of the body
  - Body frame is not moving w.r.t. to the body

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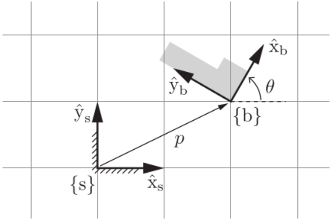

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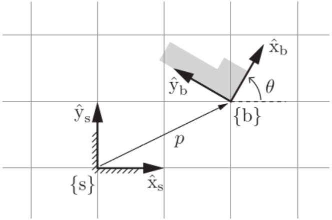

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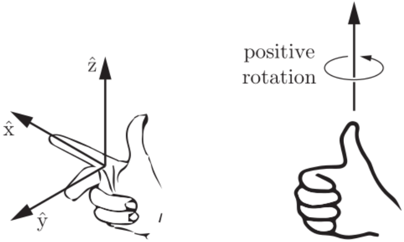
- All frames are right-handed



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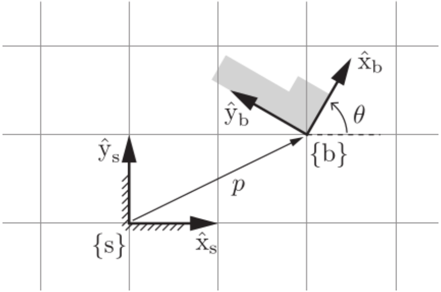

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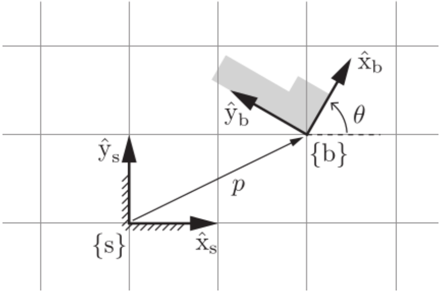

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  - position of body frame w.r.t. reference frame
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- Body frame origin
  - $\boldsymbol{p} = p_x \hat{\boldsymbol{x}}_{\boldsymbol{s}} + p_y \hat{\boldsymbol{y}}_{\boldsymbol{s}} \in \mathbb{R}^2$
  - If reference frame is clear from the context: $\boldsymbol{p} = (p_x, p_y)^\top$

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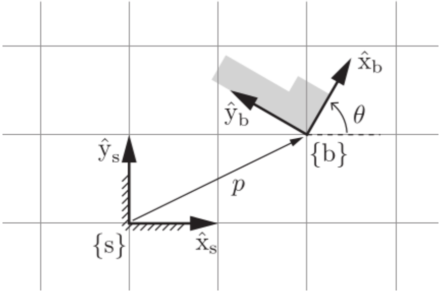

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- Orientation
  - Angle $\theta \in \langle 0^\circ, 360^\circ)$

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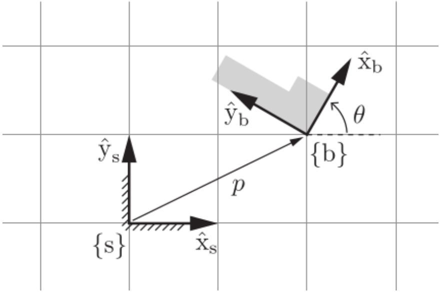

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- Orientation
  - Angle $\theta \in \langle 0^\circ, 360^\circ)$
  - Convenient for next computations:

    $\hat{\boldsymbol{x}}_{\boldsymbol{b}} = +\cos\theta \hat{\boldsymbol{x}}_{\boldsymbol{s}} + \sin\theta \hat{\boldsymbol{y}}_{\boldsymbol{s}}$

    $\hat{\boldsymbol{y}}_{\boldsymbol{b}} = -\sin\theta \hat{\boldsymbol{x}}_{\boldsymbol{s}} + \cos\theta \hat{\boldsymbol{y}}_{\boldsymbol{s}}$

    Rotation matrix $R = (\hat{\boldsymbol{x}}_{\boldsymbol{b}}, \hat{\boldsymbol{y}}_{\boldsymbol{b}}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$SO(2)$

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- Set of all rotation matrix is $SO(2)$ group, *i.e.* $R \in SO(2)$
  - Special Orthogonal group
  - $\det(R) = 1$
  - $RR^\top = I$, *i.e.* $R^{-1} = R^\top$
  - $(R_1 R_2) R_3 = R_1 (R_2 R_3)$
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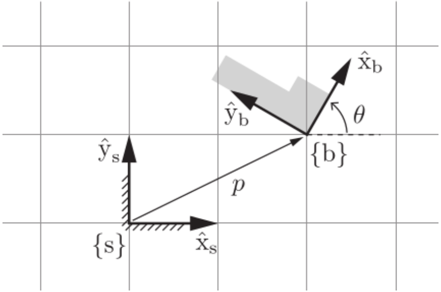
  - **For** $SO(2)$ $R_1 R_2 = R_2 R_1$
- Usage of rotation matrix
  - to represent an orientation of the frame
  - to change the reference frame in which a vector is represented
  - to rotate vector/frame

$SE(2)$

- A pair $(R_{ab}, \boldsymbol{p})$
  - represents pose/configuration of the body



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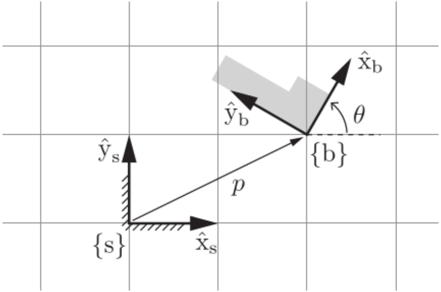

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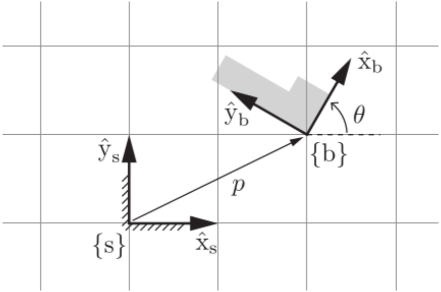

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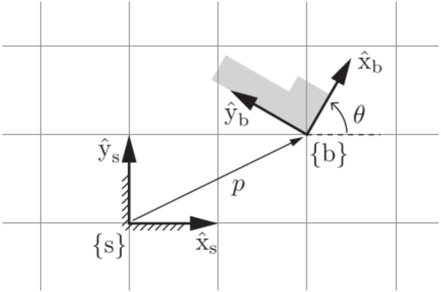

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- Alternatively, in homogeneous coordinates $T_{ab} = \begin{pmatrix} R_{ab} & \boldsymbol{p} \\ \boldsymbol{0}^\top & 1 \end{pmatrix} \in SE(2)$
  - Special Euclidean Group
  - represents both translation and rotation in a single matrix
  - $\boldsymbol{v}_a^H = T_{ab}\boldsymbol{v}_b^H$
  - $(T_1T_2)\,T_3 = T_1\,(T_2T_3)$
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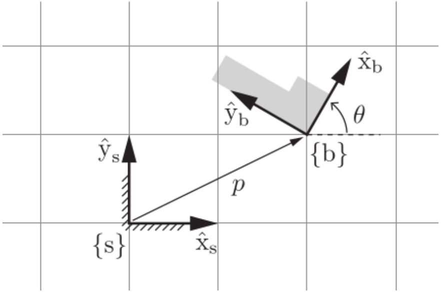

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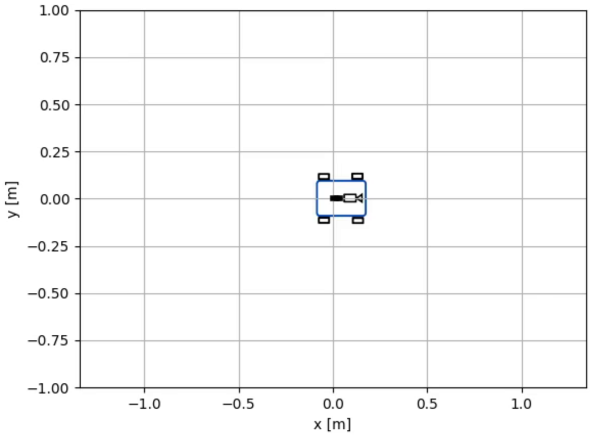
  - $T_1 T_2 \neq T_2 T_1$
  - Inverse T^{-1}
    - computing inverse of a matrix is costly
    - $T^{-1} = \begin{pmatrix} R^\top & -R^\top \boldsymbol{t} \\ \boldsymbol{0}^\top & 1 \end{pmatrix}$

$SE(2)$ example

$$T_{\text{next}} = T_{\text{current}} T_x(\delta_x) \qquad T_{\text{next}} = T_{\text{current}} T_\theta(\delta_\theta) \qquad T_{\text{next}} = T_{\text{current}} T_x(\delta_x)$$

Delta transformations are defined in robot frame.



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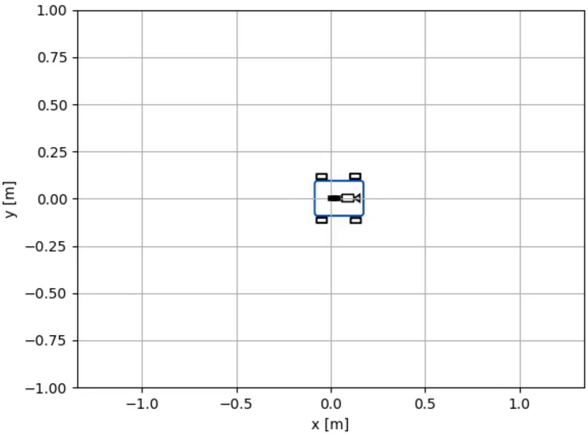
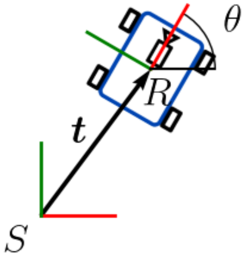


$SE(2)$ example camera

$$T_{SR} = \begin{pmatrix} R(\theta) & \boldsymbol{t} \\ \mathbf{0}^\top & 1 \end{pmatrix}$$



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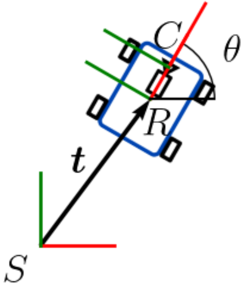

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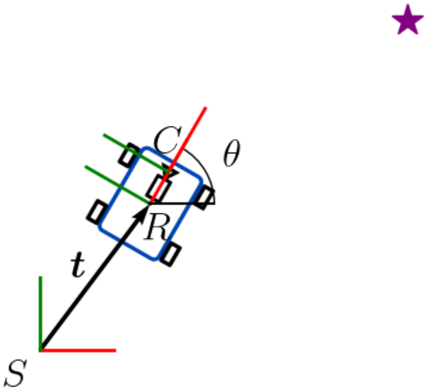

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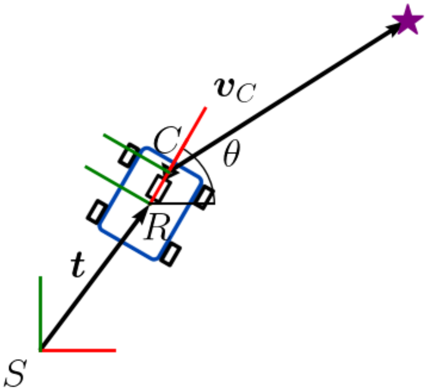

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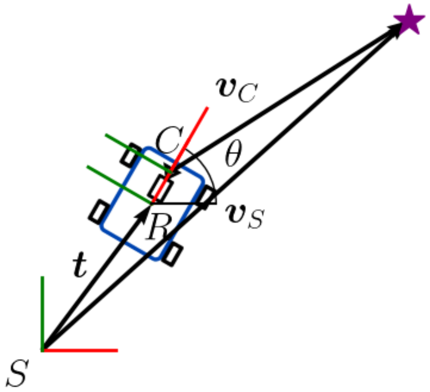
How to compute $\boldsymbol{v}_S$?



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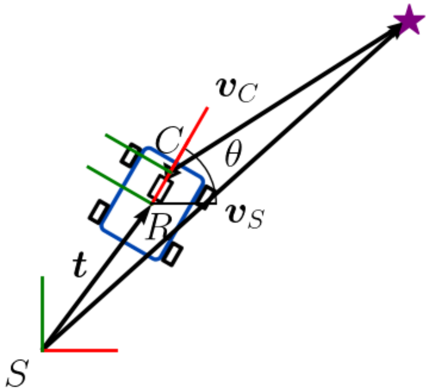
How to compute $\boldsymbol{v}_S$?

$T_{SC} = T_{SR} T_{RC}$

$\boldsymbol{v}_S = T_{SC} \boldsymbol{v}_C$



Extending to $SO(3)$ and $SE(3)$

- $SO(3)$
  - $\det(R) = 1$
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  - $T^{-1} = \begin{pmatrix} R^\top & -R^\top \boldsymbol{t} \\ \boldsymbol{0}^\top & 1 \end{pmatrix}$

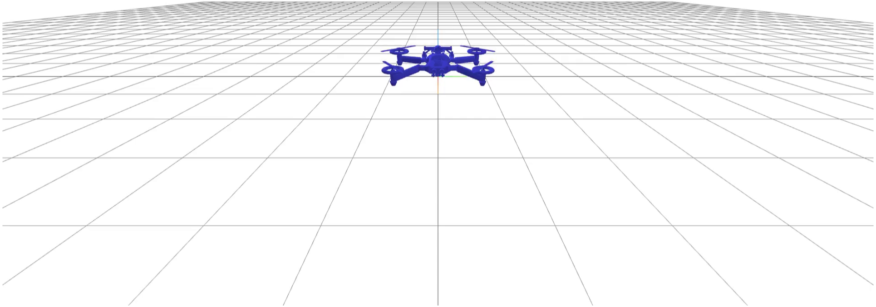
How to compute $R \in SO(3)$?

- Composing rotations around the x, y, z axes
  - $R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$
  - $R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$
  - $R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- From other representations of rotations

Example of $SE(3)$

$T_{\text{next}} = TT_z(\delta_z)$ $\quad$ $T_{\text{next}} = TR_z(\theta_z)$ $\quad$ $T_{\text{next}} = TR_y(\theta_y)$ $\quad$ $T_{\text{next}} = TT_x(\delta_x)$

$R_y, R_z \in SE(3)$!



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  - Otherwise $\theta = \arccos\left(1/2\left(\operatorname{tr} R - 1\right)\right)$ and $[\hat{\boldsymbol{\omega}}] = \frac{1}{2\sin\theta}(R - R^\top)$

Exponential coordinates

- A single vector $\boldsymbol{\omega} \in \mathbb{R}^3$
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- Why exponential?
  - it correspond to matrix exponential/logarithm of $[\boldsymbol{\omega}]$
  - if $\boldsymbol{\omega}$ is angular velocity, its integration for one unit of time leads to exponential and the final orientation is R
  - numerically sensitive to small angles

Quaternions

- $\boldsymbol{q} \in \mathbb{R}^4, \quad \|\boldsymbol{q}\| = 1$
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- Quaternions are not unique, two solutions for the same R
- Numerically stable

Other representations

- Euler angles
  - three numbers $\theta_1, \theta_2, \theta_3$
  - rotation about the x, y, or z axes
  - *e.g.* XYX Euler angles correspond to $R = R_x(\theta_1)R_y(\theta_2)R_x(\theta_3)$
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- 6D representation of rotation
  - represented by the first two columns of R
  - smooth representation
  - used in machine-learning (*e.g.* output of neural network)

Summary

- Configuration, Configuration Space $\mathcal{C}$, DoF
- Planar rigid body motion $SO(2)$, $SE(2)$
- Spatial rigid body motion $SO(3)$, $SE(3)$
- Properties of rotation matrix in $SO(2)$ and $SO(3)$
- Representation of spatial rotations
  - rotation matrix
  - axis-angle
  - exponential coordinates
  - quaternions
  - Euler angles
  - 6D representation

Laboratories goal

- Start implementing robotics toolbox
- `https://robotics-labs.readthedocs.io/`
- Utilities to work with $SO(2)$, $SE(2)$, $SO(3)$, $SE(3)$
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  - $\dots$
- Preparation
  - Linux and Conda are recommended
  - Install conda
  - Install Python IDE (PyCharm, VSCode)