Robotics: Forward kinematics of open chains

Vladimír Petrík

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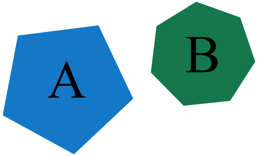
30.09.2023

Connecting two rigid bodies



- How many DoF has system of two planar rigid bodies?

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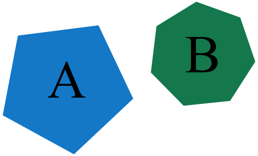
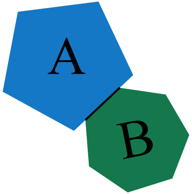


- How many DoF has system of two planar rigid bodies? $(3+3)$ DoF
- How many DoF if we glue/fix them together?



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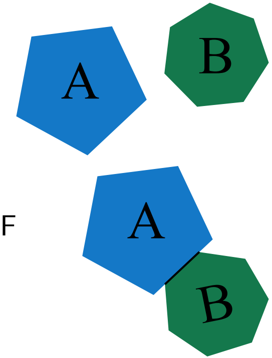
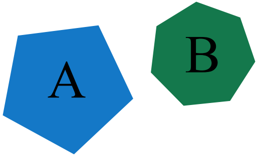
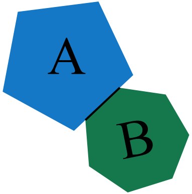


- How many DoF has system of two planar rigid bodies? $(3 + 3)$ DoF
- How many DoF if we glue/fix them together? 3 DoF
- **Fixed joint**
  - connects two rigid bodies together
  - removes 3 DoF in planar case and 6 DoF in spatial case

Connecting two rigid bodies





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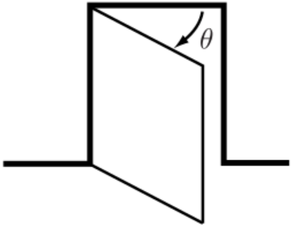
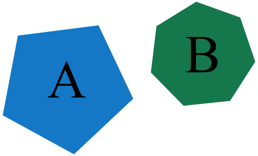
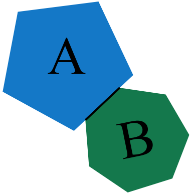


Connecting two rigid bodies





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- **Fixed joint**
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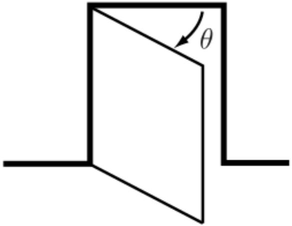
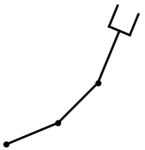
- How many DoF for door if there is no joint?
- **Revolute joint**
  - connects two rigid bodies together
  - has 1 DoF
  - removes 2 DoF in planar case and 5 DoF in spatial case



Robotic manipulator

- Constructed from links (typically rigid bodies)
- **Two** links are connected by various joints
- Actuators deliver torque/force to cause link motions
- End-effector/Gripper is attached to some of the links



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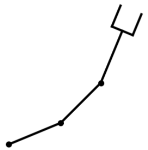

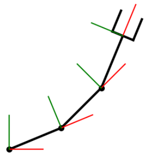

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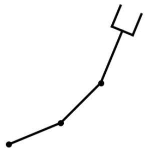

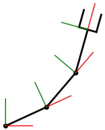

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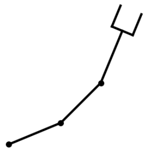
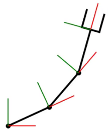
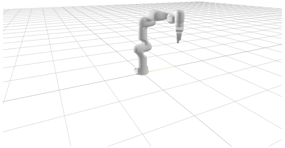
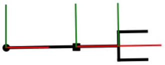




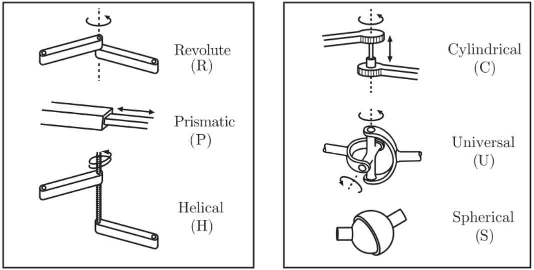


Prismatic joint

- Also sliding or linear joint
- Only translation motion in 1 DoF
- Removes 2 DoF in planar case and 5 DoF in spatial case



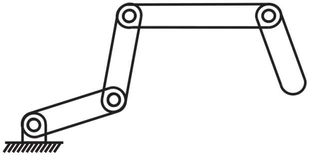
Joints types



		Constraints between two rigid bodies	
Joint type	DoF	Planar	Spatial
R	1	2	5
P	1	2	5
H	1	-	5
C	2	-	4
U	2	-	4
S	3	-	3

Open/Closed kinematic chain

- Open kinematics chains: no loops
- Closed kinematic chains contains loops



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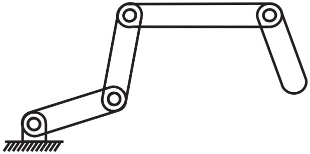
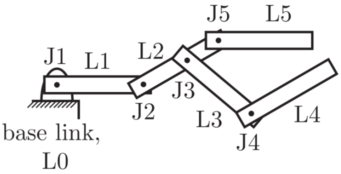


Open - sequential



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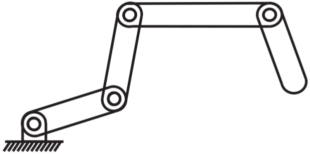

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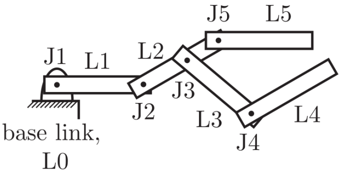
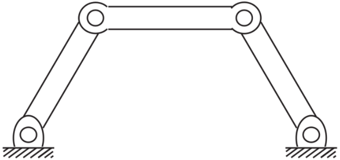


Open - tree



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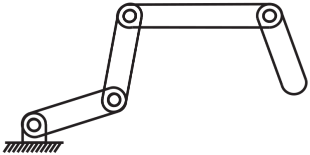

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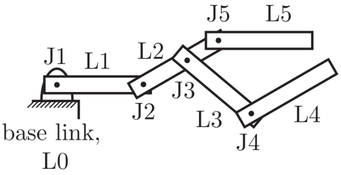

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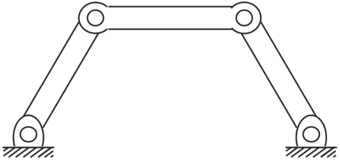
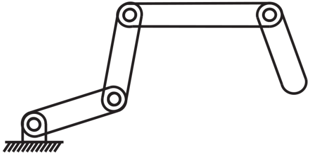


Closed

How many DoF?



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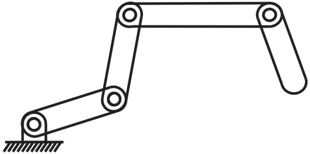
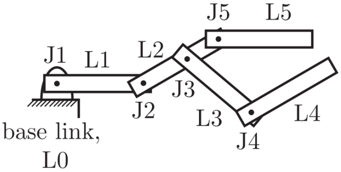


Easy: 4 DoF



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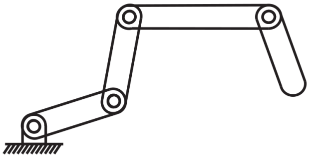

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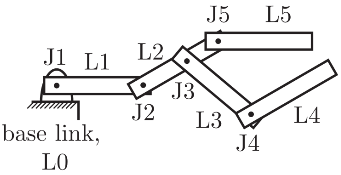
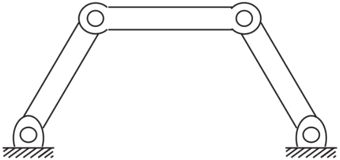


Easy: 5 DoF



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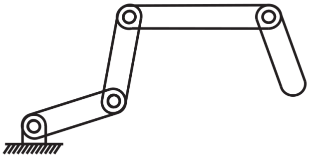

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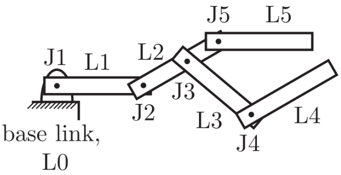

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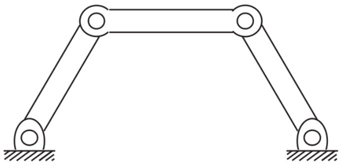

More difficult: 1 DoF

Grübler's formula

- $n_{\text{DoF}} = m\,(L-1) - \sum_{i=1}^{N} c_i$
- L is number of links including ground
- N is number joints
- m is DoF of rigid body (3 for planar, 6 for spatial)
- c_i number of constrains provided by joint i

Grübler's formula

- $n_{\mathsf{DoF}} = m\,(L-1) - \sum_{i=1}^{N} c_i = m\,(L-1-N) + \sum_{i=1}^{N} f_i$
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- c_i number of constrains provided by joint i
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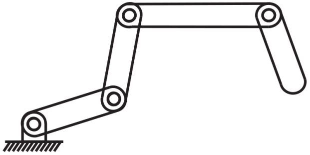
- $f_i + c_i = m$
- Works for *generic* cases, fails under certain configurations - when joints constrains are not independent

Applications of Grübler's formula

$$n_{\text{DoF}} = m\,(L - 1 - N) + \sum_{i=1}^{N} f_i$$

m - body DoF, L - number of links, N - number of joints, f_i - joint DoF



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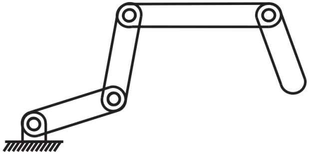

$3(5-1-4)+(1+1+1+1) =$
4 DoF

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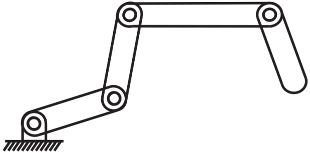
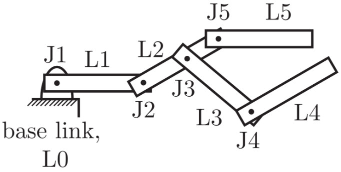


$3(5-1-4)+(1+1+1+1) =$ 4 DoF

$3(6-1-5)+(1+1+1+1+0) = 4$ DoF (if fixed joint)



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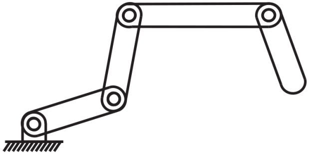

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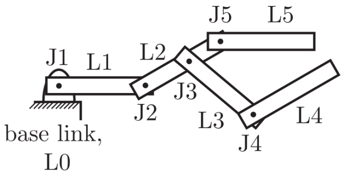
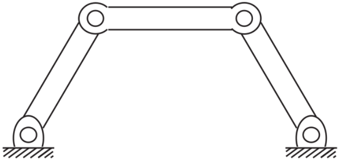


$3(6-1-5)+(1+1+1+1+1) = 5$ DoF



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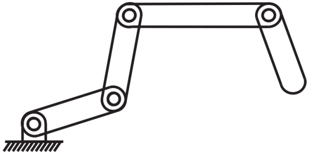

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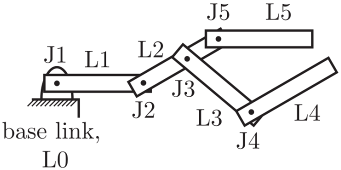

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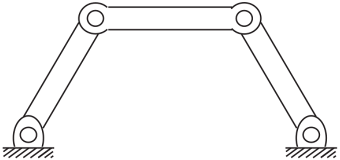

$3(4-1-4)+(1+1+1+1) =$ 1 DoF

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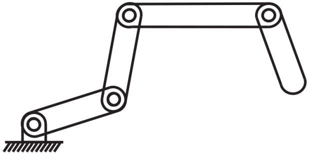

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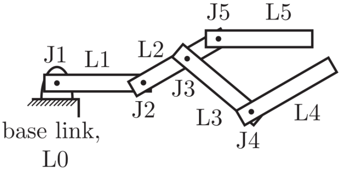

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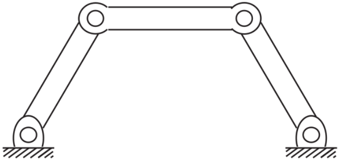
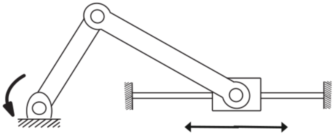


$3(4-1-4)+(1+1+1+1) =$ 1 DoF

$3(5-1-5)+(1+1+1+1+0) = 1$ DoF (if two grounds connected by fixed joint)

Application of Grübler's formula



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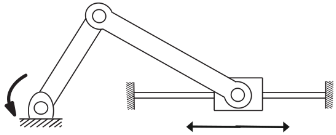

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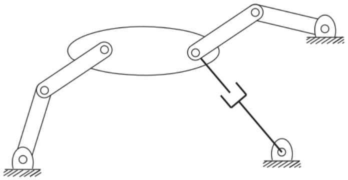

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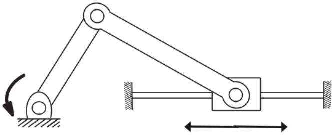

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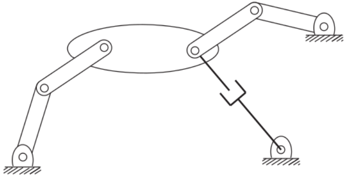
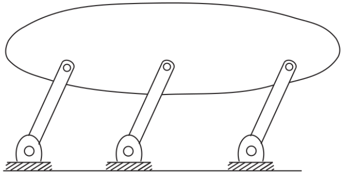
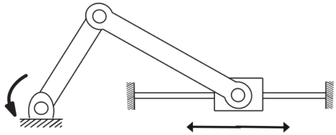


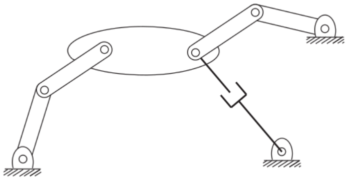
$3(8-1-9)+(9) = 3$ DoF



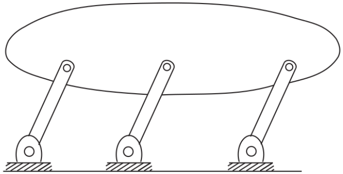
Application of Grübler's formula



$3(4-1-4)+(1+1+1+1) =$ 1 DoF



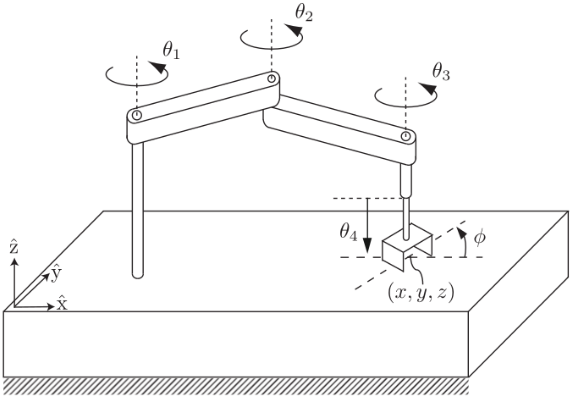
$3(8-1-9)+(9)=3$ DoF



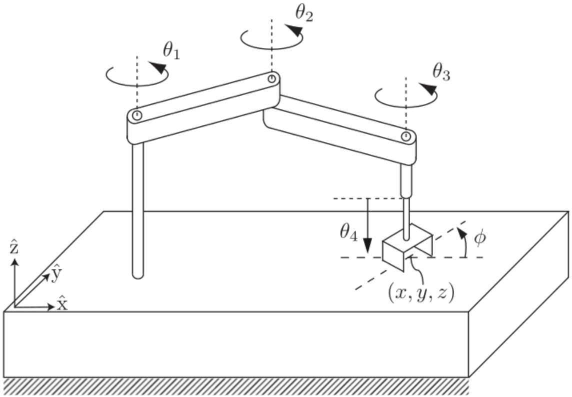
$3(5-1-6)+(6)=0$ DoF

Failure: 1 **DoF** Grübler's formula requires independent constraints provided by the joints.

Application of Grübler's formula



Application of Grübler's formula



6 $(5 - 1 - 4) + (1 + 1 + 1 + 1) = 4$ DoF

Kinematics tasks

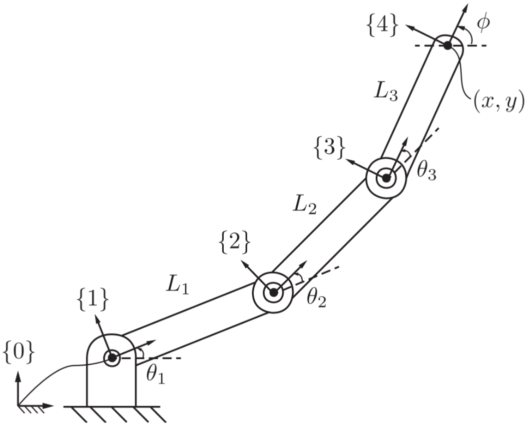
- Forward kinematics (FK)
  - calculation of the pose of the end-effector from joint coordinates
  - $f_{\mathsf{fk}} : \boldsymbol{q} \rightarrow T_{ee}$
  - $\boldsymbol{q} \in \mathbb{R}^N$, where N is number of joints
  - $T_{ee} \in SE(2)/SE(3)$

Kinematics tasks

- Forward kinematics (FK)
  - calculation of the pose of the end-effector from joint coordinates
  - $f_{\mathsf{fk}} : \boldsymbol{q} \to T_{ee}$
  - $\boldsymbol{q} \in \mathbb{R}^N$, where N is number of joints
  - $T_{ee} \in SE(2)/SE(3)$
- Inverse kinematics (IK)
  - calculation of joint coordinates from the given end-effector pose
  - $f_{\mathsf{ik}} : T_{ee} \to \boldsymbol{q}$
  - $\boldsymbol{q} \in \mathbb{R}^N$, where N is number of joints
  - $T_{ee} \in SE(2)/SE(3)$

Forward kinematics

Goal: compute FK, *i.e.* x, y, ϕ from $\boldsymbol{q} = \begin{pmatrix} \theta_1 & \theta_2 & \theta_3 \end{pmatrix}^\top$



Frame $\{0\}$ origin is located in the first joint axis of rotation.

Solution

- Trigonometry solution:

$$
\begin{aligned}
x &= L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) \\
y &= L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3) \\
\phi &= \theta_1 + \theta_2 + \theta_3
\end{aligned}
$$

  - harder to compute for spatial manipulators

Solution

- Trigonometry solution:

$$
\begin{aligned}
x &= L_1 \cos\theta_1 + L_2\cos(\theta_1+\theta_2) + L_3\cos(\theta_1+\theta_2+\theta_3)\\
y &= L_1 \sin\theta_1 + L_2\sin(\theta_1+\theta_2) + L_3\sin(\theta_1+\theta_2+\theta_3)\\
\phi &= \theta_1+\theta_2+\theta_3
\end{aligned}
$$

  - harder to compute for spatial manipulators
- Transformation based solution:

$$
\begin{aligned}
T_{04} = R(\theta_1)T_x(L_1)R(\theta_2)T_x(L_2)R(\theta_3)T_x(L_3)\\
R \in SE(2),\, T_x \in SE(2)
\end{aligned}
$$

  - more systematic solution

Solution

- Trigonometry solution:

$$
\begin{aligned}
x &= L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) \\
y &= L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3) \\
\phi &= \theta_1 + \theta_2 + \theta_3
\end{aligned}
$$

  - harder to compute for spatial manipulators
- Transformation based solution:

$$
\begin{aligned}
T_{04} = R(\theta_1)T_x(L_1)R(\theta_2)T_x(L_2)R(\theta_3)T_x(L_3) \\
R \in SE(2),\ T_x \in SE(2)
\end{aligned}
$$

  - more systematic solution
  - how to get x, y, ϕ from $T = T_{04}$?

Solution

- Trigonometry solution:

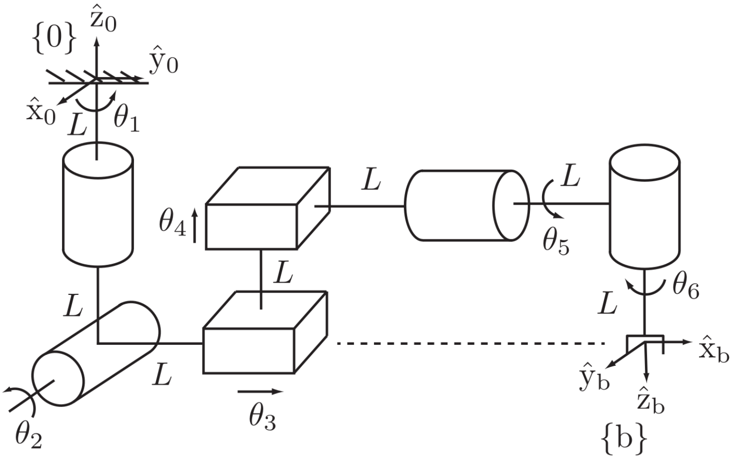
$$
\begin{aligned}
x &= L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) \\
y &= L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3) \\
\phi &= \theta_1 + \theta_2 + \theta_3
\end{aligned}
$$

  - harder to compute for spatial manipulators
- Transformation based solution:

$$
\begin{aligned}
T_{04} = R(\theta_1)T_x(L_1)R(\theta_2)T_x(L_2)R(\theta_3)T_x(L_3) \\
R \in SE(2),\ T_x \in SE(2)
\end{aligned}
$$

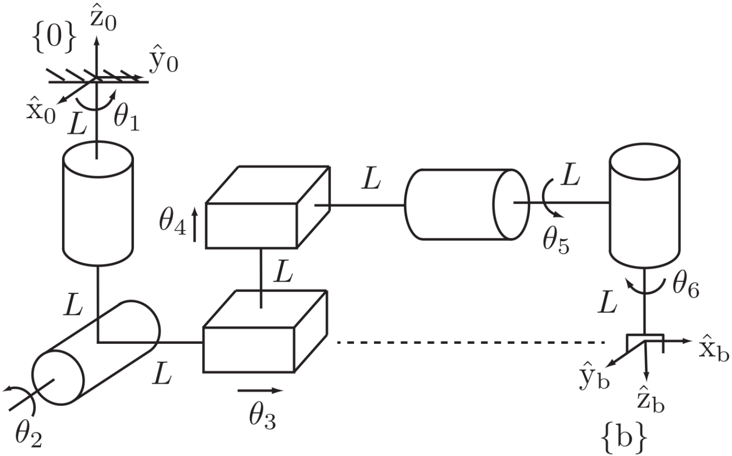
  - more systematic solution
  - how to get x, y, ϕ from $T = T_{04}$?
  - $x = T_{13}, \quad y = T_{23}, \quad \phi = \operatorname{atan2}(T_{21}, T_{22})$

Forward kinematics for spatial robot



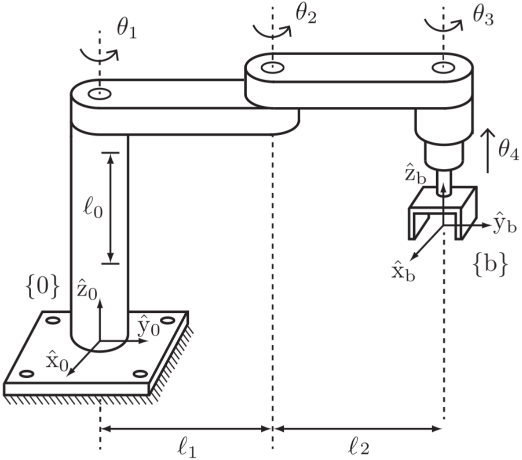
$T_{0b} = ?$

Forward kinematics for spatial robot



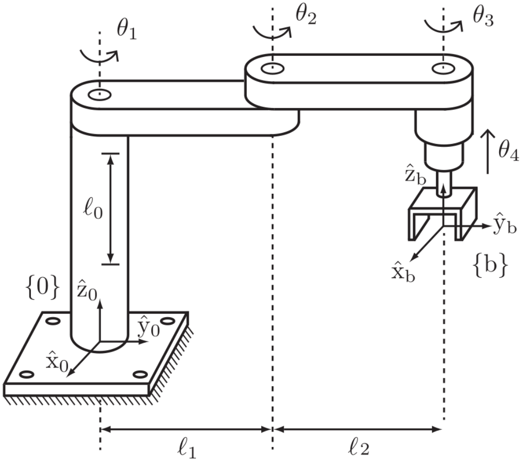
$$T_{0b} =$$
$$R_z(\theta_1)T_z(-L)R_x(\theta_2)T_y(L)T_y(\theta_3)T_z(L+\theta_4)T_y(L)R_y(\theta_5)R_z(-\theta_6)T_z(L)R_z(\pi/2)R_x(\pi)$$

Forward kinematics for spatial robot



$T_{0b} = ?$

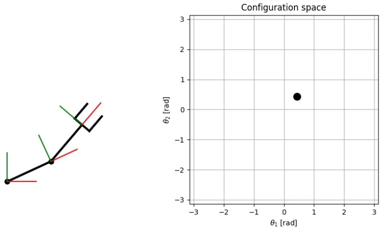
Forward kinematics for spatial robot



$$T_{0b} = T_z(l_0)R_z(\theta_1)T_y(l_1)R_z(\theta_2)T_y(l_2)R_z(\theta_3)T_z(-\theta_4)$$

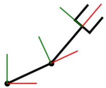
Configuration space and Task space

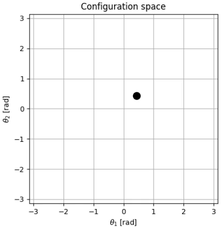
- Configuration space for 2 DoF robot
  - every point corresponds to a configuration



Configuration space and Task space

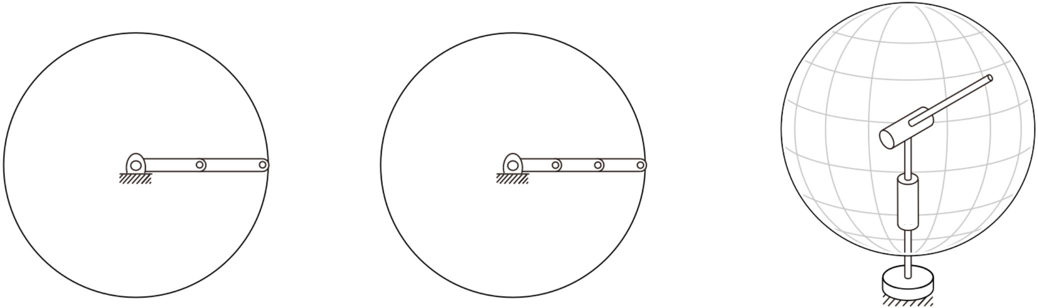
- Configuration space for 2 DoF robot
  - every point corresponds to a configuration
- Task space
  - a space in which the robot's task can be naturally expressed (robot independent)
  - a point in a task-space can be reached by multiple configurations
  - *e.g.* manipulating spatial object, task space is $SE(3)$
  - *e.g.* drawing on a paper, task space is $\mathbb{R}^2$





Workspace of the robot

- Specification of the configurations that the end-effector of the robot can reach
- Depends on the robot structure
- End-effector orientation is often ignored (but it depends on the task)

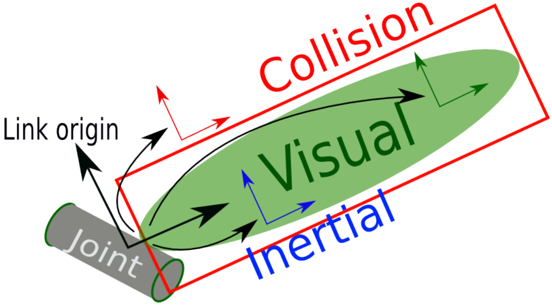


URDF

- Universal Robot Description Format
- XML file that describes robots' kinematics, geometry, dynamics
- Used in Robotic Operating System (ROS)
- Limited to open kinematic chains (including tree structures)
- Robot is described by:
  - Links (rigid body)
  - Joints (connects two links together)

URDF - Links

- Why we need visual and collision?



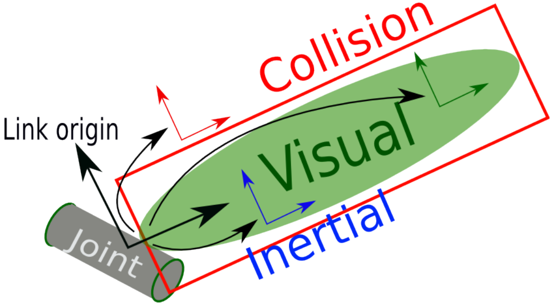

```
<robot name="robot">
    <link name="link">
        <inertial>
            ...
        </inertial>

        <visual>
            ...
        </visual>

        <collision>
            ...
        </collision>
    </link>
</robot>
```

URDF - Links

- Why we need visual and collision?
- RPY: Roll Pitch Yaw
  $R = R_z(\text{yaw})R_y(\text{pitch})R_x(\text{roll})$



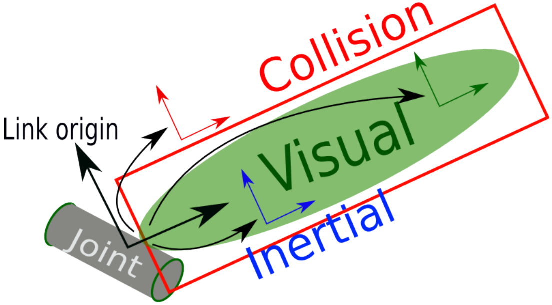

```
<robot name="robot">
    <link name="link">
        <inertial>
            <origin xyz="0 0 0.5" rpy="0 0 0"/>
            <mass value="1"/>
            <inertia ixx="100" ixy="0" ixz="0" iyy="100" iyz="0" izz="100"/>
        </inertial>

        <visual>
            ...
        </visual>

        <collision>
            ...
        </collision>
    </link>
</robot>
```

URDF - Links

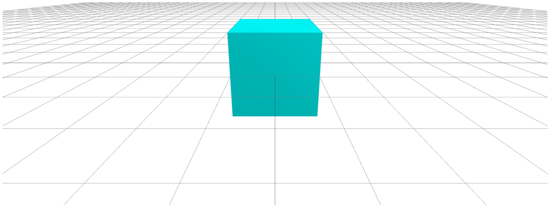
- Why we need visual and collision?
- RPY: Roll Pitch Yaw
  $R = R_z(\text{yaw})R_y(\text{pitch})R_x(\text{roll})$




```
<robot name="robot">
    <link name="link">
        <inertial>
            ...
        </inertial>

        <visual>
            <origin xyz="0 0 0" rpy="0 0 0"/>
            <geometry>
                <box size="1 1 1"/>
            </geometry>
            <material name="Cyan">
                <color rgba="0 1.0 1.0 1.0"/>
            </material>
        </visual>

        <collision>
            ...
        </collision>
    </link>
</robot>
```



URDF - Links

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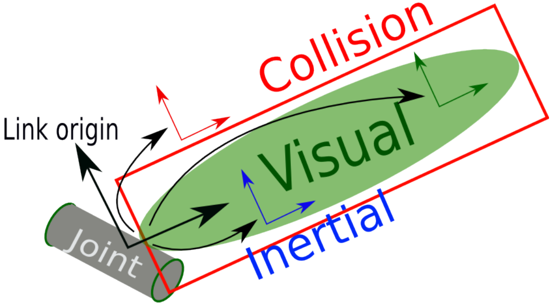
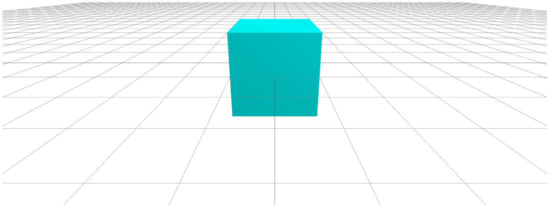
- RPY: Roll Pitch Yaw
  $R = R_z(\text{yaw})R_y(\text{pitch})R_x(\text{roll})$

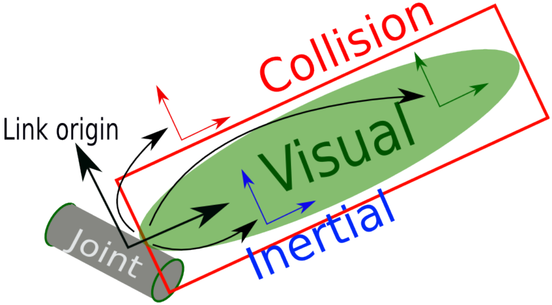



```
<robot name="robot">
    <link name="link">
        <inertial>
            ...
        </inertial>

        <visual>
            ...
        </visual>

        <collision>
            <origin xyz="0 0 0" rpy="0 0 0"/>
            <geometry>
                <cylinder radius="1" length="0.5"/>
            </geometry>
        </collision>
    </link>
</robot>
```

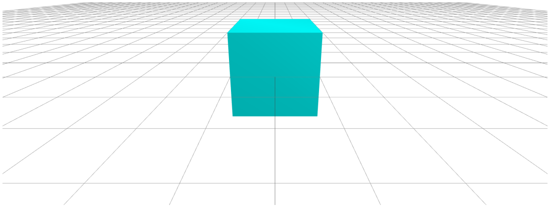


URDF - Links

- Why we need visual and collision?
- RPY: Roll Pitch Yaw
  $R = R_z(\text{yaw})R_y(\text{pitch})R_x(\text{roll})$

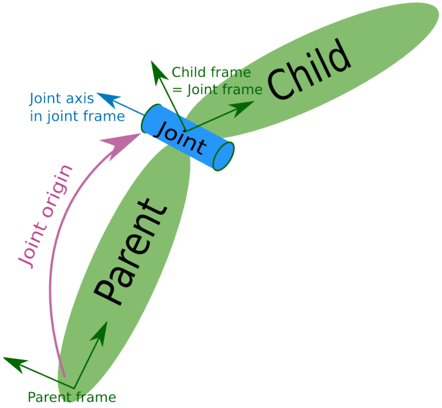



```
<robot name="robot">
    <link name="link">
        <inertial>
            ...
        </inertial>

        <visual>
            ...
        </visual>

        <collision>
            ...
        </collision>
    </link>
</robot>
```



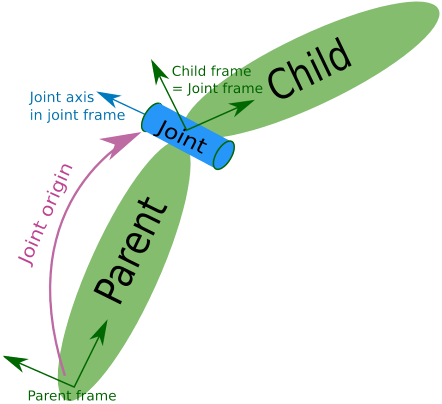
URDF - Joints

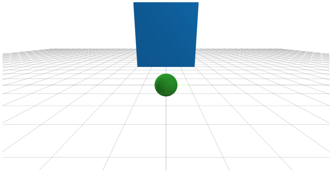
```
<link name="l0"></link> <!-- sphere -->
<link name="l1"></link> <!-- box -->
```



URDF - Joints

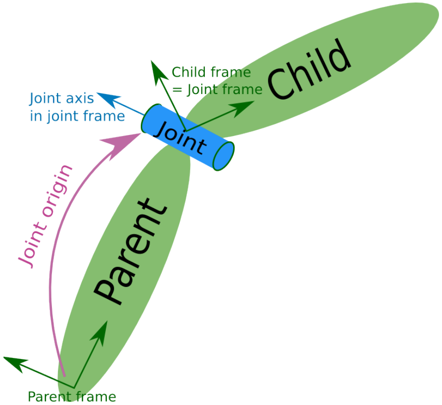
```
<link name="l0"></link> <!-- sphere -->
<link name="l1"></link> <!-- box -->

<joint name="j0" type="fixed">
   <origin xyz="0 0 1" rpy="0 0 0"/>
   <parent link="l0"/>
   <child link="l1"/>
</joint>
```

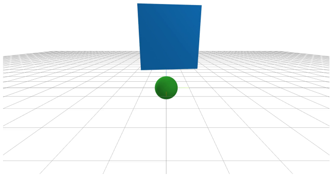




URDF - Joints

```xml
<link name="l0"></link> <!-- sphere -->
<link name="l1"></link> <!-- box -->

<joint name="j0" type="revolute">
   <origin xyz="0 0 1" rpy="0 0 0"/>
   <parent link="l0"/>
   <child link="l1"/>
   <axis xyz="0 0 1"/>
   <limit effort="30" velocity="1.0" lower="-3.14" upper="3.14" />
</joint>
```

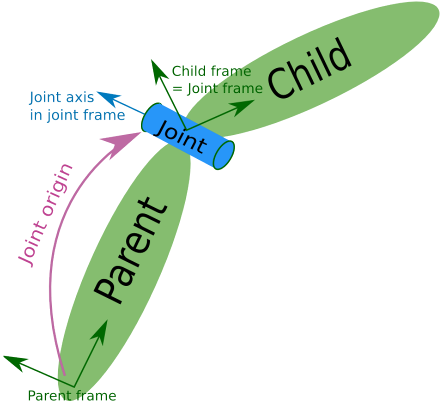


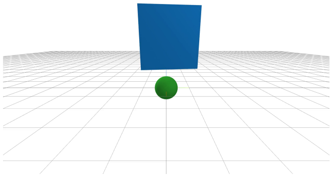


URDF - Joints

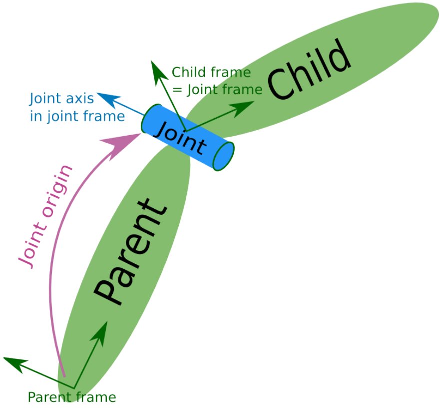
```
<link name="l0"></link> <!-- sphere -->
<link name="l1"></link> <!-- box -->

<joint name="j0" type="revolute">
   <origin xyz="0 0 1" rpy="0 0 0"/>
   <parent link="l0"/>
   <child link="l1"/>
   <axis xyz="0 0 1"/>
   <limit effort="30" velocity="1.0" lower="-3.14" upper="3.14" />
</joint>
```

- Can the upper limit be smaller than π?



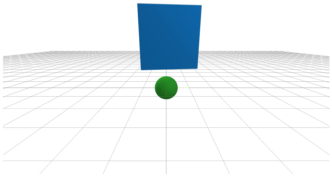


URDF - Joints

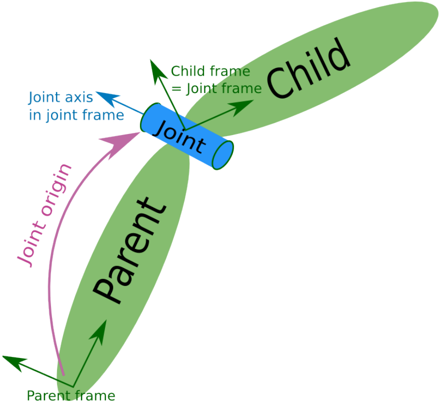



```xml
<link name="l0"></link> <!-- sphere -->
<link name="l1"></link> <!-- box -->

<joint name="j0" type="revolute">
   <origin xyz="0 0 1" rpy="0 0 0"/>
   <parent link="l0"/>
   <child link="l1"/>
   <axis xyz="0 0 1"/>
   <limit effort="30" velocity="1.0" lower="-3.14" upper="3.14" />
</joint>
```

- Can the upper limit be smaller than π?
- Can the upper limit be larger than π?

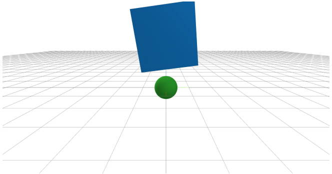


URDF - Joints

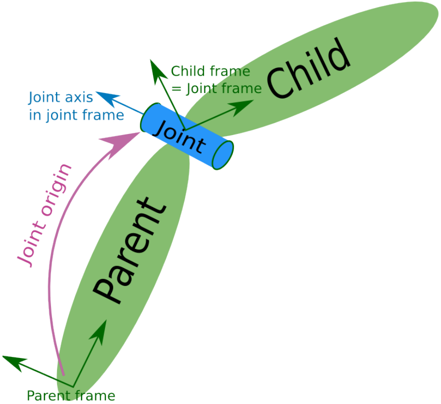



```
<link name="l0"></link> <!-- sphere -->
<link name="l1"></link> <!-- box -->

<joint name="j0" type="revolute">
   <origin xyz="0 0 1" rpy="0 0 0"/>
   <parent link="l0"/>
   <child link="l1"/>
   <axis xyz="1 0 0"/>
   <limit effort="30" velocity="1.0" lower="-3.14" upper="3.14" />
</joint>
```

- Can the upper limit be smaller than π?
- Can the upper limit be larger than π?



URDF - Joints




```
<link name="l0"></link> <!-- sphere -->
<link name="l1"></link> <!-- box -->


<joint name="joint0" type="prismatic">
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   <parent link="l0"/>
   <child link="l1"/>
   <axis xyz="0 1 0"/>
   <limit effort="30" velocity="1.0" lower="-1.0" upper="1.0" />
</joint>
```
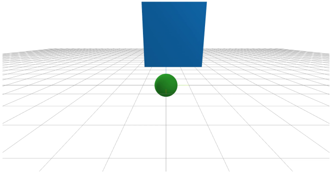
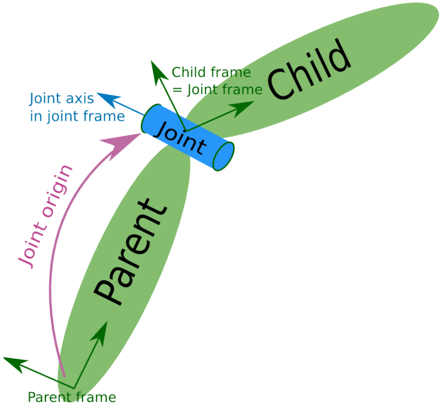

- Can the upper limit be smaller than π?
- Can the upper limit be larger than π?



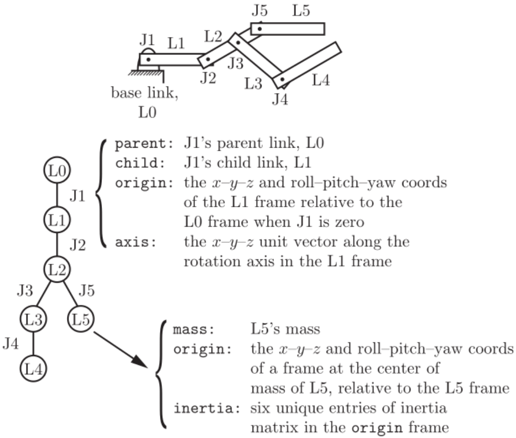
URDF - Joints




```
<link name="l0"></link> <!-- sphere -->
<link name="l1"></link> <!-- box -->
```

- Can the upper limit be smaller than π?
- Can the upper limit be larger than π?
- Other joint types: continuous, planar, floating

URDF example

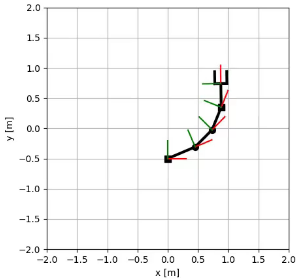


Summary

- Robotic manipulator (joints, links, end-effector)
- Joint types (DoF, constraints)
- Open/Closed kinematic chain
- Grübler's formula
- Forward/Inverse kinematics
- Configuration space / Task space / Workspace
- URDF

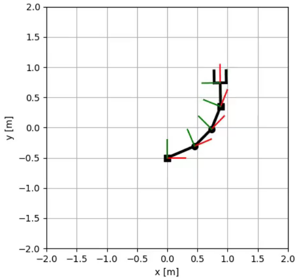
Laboratory

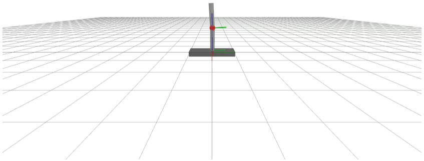
- Implement FK for planar manipulator



Laboratory

- Implement FK for planar manipulator
- Create your own URDF model





Laboratory

- Implement FK for planar manipulator
- Create your own URDF model
- Change of TA: Petr Vanc